International Journal of Theoretical Physics, Vol. 39, No. 10, 2000

Multiboson Realization of Two-Mode q-Boson Algebra with $su_q(2)$ Covariance

W.-S. Chung[1]

Received July 11, 2000

The multiboson realization of two-mode q-boson algebra with $su_q(2)$ covariance is constructed. A new quantum deformation of $su(2)$ algebra is presented by use of this realization.

Quantum groups or q-deformed Lie algebras imply specific deformations of classical Lie algebras. From a mathematical point of view, they are noncommutative associative Hopf algebras. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2].

q-Deformations of Heisenberg algebra have been made by Arik and Coon [3], Macfarlane [4], and Biedeharn [5]. Arik and Coon's q-boson algebra is written as $aa^\dagger - qa^\dagger a = 1$, while the Macfarlane–Biedenharn algebra is defined as $aa^\dagger - qa^\dagger a = q^{-N}$. Arik and Coon's q-boson algebra is somewhat different from the Macfarlane–Biedenharn's algebra in that the latter is invariant under $q \to q^{-1}$ transformation whereas the former is not. When q is real, $a^\dagger$ is interpreted as the Hermitian conjugate of a in each case. But it is not the case when the deformation parmeter q is a root of unity. In this case $a^\dagger$ is interpreted as the Hermitian conjugate of a in the Macfarlane–Biedenharn case, while it is not in the Arik–Coon case.

The multimode generalization for Macfarlane–Biedenharn q-boson algebra was discussed by Greenberg [6],

$$a_i a_i^\dagger - q a_i^\dagger a_i = q^{-N_i}$$

But this algebra is not covariant under some quantum algebra. In 1989, Pusz

[1] Theory Group, Department of Physics and Research Institute of Natural Science, Gyeongsang National Umversity, Jinju, 660-701, Korea.

0020-7748/00/1000-2407$18.00/0

and Woronowicz [7] considered the multimode generalization of Arik and Coon's single-mode q-boson algebra, which is covariant under the quantum group $su_q(n)$.

In this paper we restrict our concern to the two-mode q-boson case, which has the $su_q(2)$ covariance. Using this algebra, we construct the multiboson realization of the two-mode q-boson algebra and present a new quantum deformation of classical $su(2)$ algebra. In single-mode case [8], the multiboson realization of q-boson algebra does not give rise to the new quantum deformation of $su(2)$ algebra.

The two-mode q-boson algebra with $su_q(2)$ covariance is defined as

$$
\begin{aligned}
a_1^\dagger a_2^\dagger &= \sqrt{q}\, a_2^\dagger a_1^\dagger \\
a_1 a_2 &= \frac{1}{\sqrt{q}}\, a_2 a_1 \\
a_1 a_2^\dagger &= \sqrt{q}\, a_2^\dagger a_1 \\
a_2 a_1^\dagger &= \sqrt{q}\, a_1^\dagger a_2 \\
a_1 a_1^\dagger &= 1 + q a_1^\dagger a_1 + (q-1) a_2^\dagger a_2 \\
a_2 a_2^\dagger &= 1 + q a_2^\dagger a_2
\end{aligned}
\tag{1}
$$

Throughout, $(\cdot)^\dagger$ denotes the hermitian conjugate of $(\cdot)$. An $su_q(2)$-matrix can be written in the form

$$
M = \begin{pmatrix} a & (1/\sqrt{q}) \\ -b^* & a^* \end{pmatrix}
$$

where the following commutation relations hold:

$$
\begin{gathered}
aa^* - a^*a = \left(1 - \frac{1}{q}\right) bb^* \\
ab = \frac{1}{\sqrt{q}}\, ba, \qquad b^*a^* = \frac{1}{\sqrt{q}}\, a^*b^* \\
ab^* = \frac{1}{\sqrt{q}}\, b^*a, \qquad ba^* = \frac{1}{\sqrt{q}}\, a^*b \\
bb^* = b^*b \qquad det_q\, M = aa^* + \frac{1}{q}\, bb^* = 1
\end{gathered}
\tag{2}
$$

By the $su_q(2)$ covariance of the system, it is meant that the linear transformations

$$\begin{pmatrix} a & (1/\sqrt{q})b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} \tag{3}$$

$$(a_1^\dagger a_2^\dagger) \begin{pmatrix} a^* & -b \\ (1/\sqrt{q})b^* & a \end{pmatrix} = ((a_1^\dagger)'(a_2^\dagger)')$$

lead to the same commutation relations (1) for $(a_1^\dagger, (a_1^\dagger)')$ and $(a_2', (a_2^\dagger)')$. It should be noted that the particular coupling between the two modes is completely dictated by the required $su_q(2)$ covariance.

The Fock space representation of the algebra (1) can be easily constructed by introducing the Hermitian number operators $\{N_1, N_2\}$ obeying

$$[N_i, a_j] = -\delta_{ij} a_j, \qquad [N_i, a_j^\dagger] = -\delta_{ij} a_j^\dagger \qquad (i, j = 1, 2) \tag{4}$$

Let $|0, 0\rangle$ be the unique ground state of this system satisfying

$$N_i|0, 0\rangle = 0, \qquad a_i|0, 0\rangle = 0 \qquad (i = 1, 2) \tag{5}$$

and $\{|n, m>|n, m = 0, 1, 2, \ldots\}$ be the set of the orthogonal number eigenstates

$$N_1|n, m\rangle = n|n, m\rangle, \qquad N_2|n, m\rangle = m|n, m\rangle$$
$$\langle n, m|n', m'\rangle = \delta_{nn'}\delta_{mm'} \tag{6}$$

From the algebra (1) the representation is given by

$$a_1|n, m\rangle = \sqrt{q^m[n]}\,|n-1, m\rangle, \qquad a_2|n, m\rangle = \sqrt{[m]}\,|n, m-1\rangle$$
$$a_1^\dagger|n, m\rangle = \sqrt{q^m[n+1]}\,|n+1, m\rangle, \qquad a_2^\dagger|n, m\rangle = \sqrt{[m+1]}|n, m+1\rangle \tag{7}$$

where the q-number $[x]$ is defined as

$$[x] = \frac{q^x - 1}{q - 1}$$

The general eigenstate $|n, m\rangle$ is obtained by applying $a_2^\dagger$ m times after applying $a_1^\dagger$ n times,

$$|n, m\rangle = \frac{(a_2^\dagger)^m (a_1^\dagger)^n}{\sqrt{[n]![m]!}}\,|0, 0\rangle \tag{8}$$

where

$$[n]! = [n][n-1]\cdots[2][1], \qquad [0]! = 1$$

Now we will discuss the multiboson realization of the two-mode q-boson algebra with $su_q(2)$ covariance. To begin with, we introduce generalized step operators A_i and $A_i^\dagger$ $(i = 1, 2)$, where

$$A_i = a_i^k f_i(N_1, N_2)$$
$$A_i^\dagger = f_i(N_1, N_2)(a_i^\dagger)^k \tag{9}$$

The real function $f_i(N_1, N_2)$ should be determined in such a way that A_i, $A_i^\dagger$ satisfy a relation of the form of Eq. (1). From the relation $A_1 A_2 = (1/\sqrt{q}) A_2 A_1$, we have

$$f_1(N_1, N_2 - k) f_2(N_1, N_2) = (\sqrt{q})^{k^2-2} f_1(N_1, N_2) f_2(N_1 - k, N_2) \tag{10}$$

From the relation $A_1 A_2^\dagger = \sqrt{q} A_2^\dagger A_1$, we also have

$$f_1(N_1, N_2 + k) f_2(N_1, N_2 + k)$$
$$= \left(\frac{1}{\sqrt{q}}\right)^{k^2-1} f_1(N_1, N_2) f_2(N_1 - k, N_2 + k) \tag{11}$$

The solution of Eqs. (10) and (11) takes the form

$$f_1(N_1, N_2) = q^{(1-k^2)/2k} f(N_1), \qquad f_2(N_1, N_2) = g(N_2) \tag{12}$$

Inserting Eqs. (9) and (12) into the fifth and sixth relations of (1) gives the recurrence relation for $f(N_1)$ and $g(N_2)$ as

$$g(N_2 + k)^2 \frac{[N_2 + k]!}{[N_2]!} - q g(N_2)^2 \frac{[N_2]!}{[N_2 - k]!} = 1 \tag{13}$$

and

$$q^{N_2/k} f(N_1 + k)^2 \frac{[N_1 + k]!}{[N_1]!} - q^{N_2/k+1} f(N_1)^2 \frac{[N_1]!}{[N_1 - k]!}$$
$$= 1 + (q - 1) g(N_2)^2 \frac{[N_2]!}{[N_2 - k]!} \tag{14}$$

where we used the relations

$$(a_1^\dagger)^k a_1^k = q^{kN_2} \frac{[N_1]!}{[N_1 - k]!}$$

$$a_1^k (a_1^\dagger)^k = q^{kN_2} \frac{[N_1 + k]!}{[N_1]!}$$

$$(a_2^\dagger)^k a_2^k = \frac{[N_2]!}{[N_2 - k]!}$$

$$a_2^k (a_2^\dagger)^k = \frac{[N_2 + k]!}{[N_2]!}$$

Solving Eq. (13) for $g(N_2)$, we have

$$g(N_2) = \sqrt{\left[\left\langle \frac{N_2}{k} \right\rangle\right] \frac{[N_1 - k]!}{[N_2]!}} \tag{15}$$

where $\langle x \rangle$ is defined as the greatest integer less than or equal to x. Inserting the solution (15) into Eq. (14), we see that the right-hand side becomes $q^{\langle N_2/k \rangle}$, The left-hand side of Eq. (14) is proportional to $q^{N_2/k}$, while the right-hand side is $q^{\langle N_2/k \rangle}$. So, unless $\langle N_2/k \rangle$ is the same as N_2/k, the solution of Eq. (14) does not exist, which implies that the eigenvalue of N_2 should be k times integers. In this case, the solution of Eq. (14) becomes

$$f(N_1) = \sqrt{\left[\left\langle \frac{N_1}{k} \right\rangle\right] \frac{[N_1 - k]!}{[N_1]!}} \tag{16}$$

Thus the generalized annihilation operators become

$$\begin{aligned} A_1 &= q^{(1-k^2)/2k} a_1^k \sqrt{\left[\left\langle \frac{N_1}{k} \right\rangle\right] \frac{[N_1 - k]!}{[N_1]!}} \\ A_2 &= a_2^k \sqrt{\left[\left\langle \frac{N_2}{k} \right\rangle\right] \frac{[N_2 - k]!}{[N_2]!}} \end{aligned} \tag{17}$$

The relations between generalized step operators and number operators are

$$\begin{aligned} A_1^\dagger A_1 &= q^{N_2/k}[\langle N_1/k \rangle] \\ A_2^\dagger A_2 &= [\langle N_1/k \rangle] \end{aligned} \tag{18}$$

In this case $\langle N_1/k \rangle$ plays the role of the number operators for $A_i^\dagger$ and A_i,

$$\begin{aligned} [\langle N_1/k \rangle, A_j^\dagger] &= \delta_{ij} A_j^\dagger \\ [\langle N_1/k \rangle, A_j] &= -\delta_{ij} A_j \end{aligned} \tag{19}$$

The Fock basis for the generalized two-mode q-boson algebra is given by $|kn + p, km\rangle$, where n, m are nonnegative integers and $p = 0, 1, \ldots, k - 1$. Acting with the generalized two-mode q-boson operators on these bases, we get

$$\begin{aligned} \langle N_1/k \rangle |kn + p, km\rangle &= n|kn + p, km\rangle \\ \langle N_2/k \rangle |kn + p, km\rangle &= m|kn + p, km\rangle \\ A_1 |kn + p, km\rangle &= \sqrt{q^m[n]}\,|k(n - 1) + p, km\rangle \\ A_1^\dagger |kn + p, km\rangle &= \sqrt{q^m[n + 1]}\,|k(n + 1) + p, km\rangle \\ A_2 |kn + p, km\rangle &= \sqrt{[m]}\,|kn + p, k(m - 1)\rangle \end{aligned} \tag{20}$$

$$A_2^\dagger|kn + p, k\rangle = \sqrt{[m + 1]}|kn + p, km(m + 1)\rangle$$

This representation does not depend on k and p. For fixed k, there exist k sectors where the representations are the same. In other words, the Fock space of the two-mode q-boson algebra is divided into k sectors and the generalized two-mode q-boson algebra has the same representation in each sector. The ground state of the generalized two mode q-boson algebra in the p-sector is given by $|p, 0\rangle$, where $p = 0, 1, 2, \ldots, k - 1$. The excited state in the p-sector is obtained by applying the generalized creation operators to the p-sector ground state successively,

$$|kn + p, km\rangle = \frac{(A_2^\dagger)^m (A_1^\dagger)^n}{\sqrt{[m]![n]!}} |p, 0\rangle$$

If we construct the operators of the q-deformed $su(2)$ algebra by using the Jordan–Schwinger realization as follows,

$$\begin{aligned} J_+ &= A_1^\dagger A_2 \\ J_- &= A_2^\dagger A_1 \\ J_0 &= \frac{1}{2}(\langle N_1/k\rangle - \langle N_2/k\rangle) \end{aligned} \tag{21}$$

then we have

$$\begin{aligned} J_+J_- - qJ_-J_+ &= q^{2\langle N_2/k\rangle}[2J_0] \\ [J_0, J_\pm] &= \pm J_\pm \end{aligned} \tag{22}$$

If we replace

$$\begin{aligned} \mathcal{T}_+ &= q^{-\langle N_2/k\rangle} J_+ \\ \mathcal{T}_- &= J_- q^{-\langle N_2/k\rangle} \\ \mathcal{T}_0 &= J_0 \end{aligned}$$

the algebra (22) becomes

$$\begin{aligned} \mathcal{T}_+\mathcal{T}_- - q^{-1}\mathcal{T}_-\mathcal{T}_+ &= [2\mathcal{T}_0] \\ [\mathcal{T}_0, \mathcal{T}_\pm] &= \pm\mathcal{T}_\pm \end{aligned} \tag{23}$$

This algebra is different from the standard quantum deformation of $su(2)$ algebra. But this algebra (23) is also a quantum group whose comultiplication is defined as

$$\begin{aligned} \Delta(\mathcal{T}_\pm) &= \mathcal{T}_\pm \otimes I + q^{\mathcal{T}_0} \otimes \mathcal{T}_\pm \\ \Delta(\mathcal{T}_0) &= \mathcal{T}_0 \otimes I + I \otimes \mathcal{T}_0 \end{aligned} \tag{24}$$

To conclude, I discussed the two-mode generalization of the single-mode q-boson algebra introduced by Arik and Coon so that it may have $su_q(2)$ covariance. Using this algebra, I constructed the multiboson realization of the two-mode q-boson algebra and discussed its representation. I found that the Jordan–Schwinger realization in terms of the generalized two-mode q-boson algebra gives rise to a new quantum deformation of the $su(2)$ algebra whose comultiplication is well defined.

REFERENCES

1. M. Jimbo (1985), *Lett. Math. Phys.* **10**, 63; (1986), **11**, 247.
2. V. Drinfeld (1986), In *Proceedings of the International Congress of Mathematicians* (Berkeley), p. 78.
3. M. Arik and D. Coon (1976), *J. Math. Phys.* **17**, 524.
4. A. Macfarlane (1989), *J. Phys. A* **22**, 4581.
5. L. Biedenharn (1989), *J. Phys. A* **22**, L873.
6. O. Greenberg (1991), *Phys. Rev. D* **43**, 4111.
7. W. Pusz and S. Woronowicz (1989), *Rep. Math. Phys.* **27**, 231.
8. J. Katriel and A. Solomon (1991), *J. Phys. A* **24**, 2093.